SPACECRAFT PROXIMITY OPERATIONS

Relative Motion Modeling, Stability Analysis, Optimal Transfer Design and LQR Docking Control in the Clohessy-Wiltshire Framework

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ABSTRACT

This paper documents the design and numerical validation of a complete spacecraft proximity operations pipeline for a chaser vehicle approaching a target in low Earth orbit. The work proceeds from first principles through the Clohessy-Wiltshire linearization of the Hill equations, derives the full eigenstructure of the relative motion dynamics, and demonstrates that the system is marginally stable with a double zero eigenvalue responsible for secular along-track drift. A five-scenario simulation campaign quantifies the accuracy of the linear approximation against the full nonlinear model. An analytical closed-form solution is validated for all scenarios. A globally optimal two-impulse transfer maneuver is designed using the CW state-transition matrix and a three-parameter numerical optimization, comparing four solvers to identify the true global minimum. A Linear Quadratic Regulator with Bryson-rule weighting and scalar rho optimization drives the final docking approach on the nonlinear plant. The complete four-phase mission is integrated in Simulink. A global co-optimization simultaneously tunes the transfer geometry and the LQR weight, reducing total mission delta-V by 24 percent relative to the sequentially designed baseline.

Keywords: *proximity operations / Clohessy-Wiltshire equations / orbital rendezvous / LQR control / state-transition matrix / global optimization*

COURSE

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Laboratory 02 — Proximity Operations

CONTENTS

01 INTRODUCTION AND MISSION OVERVIEW

Spacecraft proximity operations — the coordinated maneuvering of a chaser vehicle into close range of an orbiting target — represent one of the most demanding and precisely constrained problems in astrodynamics. The applications span orbital servicing, debris removal, crew transfer, and autonomous docking of supply vehicles to space stations. In every case, the chaser must navigate a dynamic relative-motion environment shaped by gravitational gradients, Coriolis effects, and the geometry of the rotating reference frame, while respecting hard constraints on propellant expenditure and terminal approach safety.

This paper presents a complete engineering treatment of the proximity operations problem, developed collaboratively by the present author and colleagues as part of Laboratory 02 of the course Analisi e Simulazione di Sistemi Aerospaziali at Politecnico di Milano, academic year 2025/2026. The study proceeds in eight connected stages: nonlinear simulation of relative motion in five representative initial-condition scenarios; linearization via the Clohessy-Wiltshire (CW) equations [2]; modal and stability analysis of the linearized system; numerical comparison of the linear and nonlinear models; derivation and validation of the analytical closed-form solution; globally optimal two-impulse transfer design via the CW state-transition matrix (STM); Linear Quadratic Regulator (LQR) design for the final docking approach; and a global co-optimization that simultaneously tunes the transfer geometry and the controller weight to minimize the total mission delta-V.

The reference target orbit is a circular low Earth orbit at altitude 500 km. The chaser is treated as a point mass; atmospheric drag, solar radiation pressure, and Earth oblateness (J2) are neglected throughout, consistent with the short-duration proximity operations regime where these perturbations accumulate slowly relative to the maneuver timescale. The Clohessy-Wiltshire linearization is valid as long as the chaser-target separation remains small relative to the orbital radius; the present simulations cover separations up to 15.8 km, and the error analysis in Section 06 quantifies where this assumption begins to degrade.

The complete pipeline from orbital mechanics to optimal control is implemented in MATLAB, with the Simulink model used for the full four-phase mission integration. A shared database structure holds all orbital constants and initial conditions, and a single ODE right-hand side function switches between the nonlinear and linearized dynamics via a flag argument, ensuring that both models are propagated under identical numerical conditions.

02 REFERENCE FRAME AND ORBITAL PARAMETERS

The analysis is conducted in the Local-Vertical Local-Horizontal (LVLH) frame, also known as the Hill frame [3], centered on the target spacecraft. The x-axis points radially outward from the Earth center, the y-axis is along the target velocity vector (along-track), and the z-axis completes a right-handed triad pointing out of the orbital plane. This coordinate choice is standard in proximity operations and rendezvous analysis because it makes the orbital mechanics transparent: bounded in-plane motion appears as ellipses, secular drift is immediately visible as linear growth in the y-direction, and out-of-plane motion decouples entirely.

The target occupies a circular orbit at altitude $h_0 = 500$ km above a spherical Earth. The relevant orbital parameters are summarized below.

Parameter	Symbol	Value
Gravitational parameter	μ	398600.44
Earth radius	R_a	6378.14
Orbit altitude	h_0	500
Orbital radius	$R_0 = R_a + h_0$	6878.14
Mean motion	$n = \sqrt{(\mu/R_0^3)}$	1.1067×10^{-3}
Orbital period	$T = 2\pi/n$	5676

Table 1 — Orbital parameters for the target spacecraft at 500 km altitude.

The five initial-condition scenarios used throughout the study are labeled A through E. Scenarios A and E satisfy the CW closed-orbit condition $\dot{y}_0 = -2nx_0$, ensuring bounded in-plane motion. Scenarios B and D have $\dot{y}_0 = -1.5nx_0$, introducing a secular drift. Scenario C places the chaser at a pure along-track offset with no radial displacement. Scenario E uses a large initial separation of 15 km radially and 5 km out of plane, providing a stress test for the linearization.

03 NONLINEAR RELATIVE MOTION MODEL

3.1 Equations of Motion in the LVLH Frame

The exact equations governing the relative position of the chaser with respect to the target in the LVLH frame are derived from the two-body equations of motion for each spacecraft, subtracting and projecting onto the rotating frame. The result is a set of three coupled nonlinear ODEs:

$$\ddot{x}^{\cdot\cdot} = 2n\dot{y} + n^2(R_0 + x) - \mu(R_0 + x) / r^{c\,3}$$

$$\tilde{y}^{\cdot\cdot} = -2n\dot{x} + n^2 y - \mu y / r^{c\,3}$$

$$\underset{\cdot}{z}^{\cdot\cdot} = -\mu z / r^{c\,3}$$

$$r^c = \sqrt{}[(R_0 + x)^2 + y^2 + z^2]$$

The 2n terms are Coriolis accelerations coupling the radial and along-track directions. The n^2 terms arise from the centrifugal contribution of the rotating reference frame. The r^{c-3} terms represent the full two-body gravitational attraction on the chaser. No linearizing assumptions have been made; these equations are exact within the two-body problem.

3.2 Numerical Implementation

The MATLAB implementation encodes these equations inside the function odefun, which accepts the six-component state vector $X = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$ and a flag argument selecting between the full nonlinear model (flag = 0) and the linearized CW model (flag = 1). This single-function design ensures that the two models are integrated under identical numerical conditions and that the error analysis in Section 06 is not contaminated by implementation differences.

Numerical integration uses MATLAB’s ode45 solver (Dormand-Prince explicit Runge-Kutta 4/5) with relative tolerance 10^{-6} and absolute tolerance 10^{-8}. Each scenario is propagated over two complete orbital periods $t \in [0, 2T]$, producing a time vector and a 6-column state matrix. The orbital period T and mean motion n are computed once from the database structure and reused throughout.

```
[t{k}, res{k}] = ode45(@(t,X) odefun(X,
db, flag), [0 2*T], x0, opts);
```

Fig. 1 — Core integration call. The anonymous function closure captures db and flag, keeping the ODE interface stateless.

3.3 Multi-Scenario Simulation Results

The five scenarios produce qualitatively distinct trajectories that collectively illustrate the full range of CW free-motion behavior. Scenario A traces a regular closed 2:1 ellipse in the x-y plane because its initial velocity satisfies the closed-orbit condition; the out-of-plane component oscillates independently at frequency n. Scenario B has a radial offset with insufficient tangential velocity: the oscillatory x and y terms cancel, leaving the chaser at a constant radial offset while drifting backward along the -y direction. Scenario C places the chaser at a pure along-track offset with no radial displacement or velocity; it remains at a constant x = 0 and $y = y_0$ while oscillating out-of-plane. Scenario D combines the in-plane behavior of B with an independent out-of-plane oscillation from a non-zero z_0. Scenario E is geometrically identical to A but at 150 times the separation scale, exercising the large-offset regime where the linearization begins to degrade.

The nonlinear trajectories confirm physically expected behavior and serve as the reference against which the linearized model is evaluated. Position and velocity plots for all five scenarios are generated by the function plot_trajectories, which produces 3D figures in the LVLH frame with the target marked at the origin.

04 LINEARIZATION: THE CLOHESSY-WILTSHIRE EQUATIONS

4.1 Derivation of the CW Model

The linearized equations are derived under the assumption that the chaser-target separation is small relative to the orbital radius, $||r||/R_0 \ll 1$. Under this condition $r^c \approx R_0 + x$, and the gravitational terms expand to first order in x/R_0. Using $n^2 = \mu/R_0^3$ and canceling the constant terms that describe the target orbit itself, the Clohessy-Wiltshire equations [2] are obtained:

$$\ddot{x} = 2n\dot{y} + 3n^2x$$

$$\ddot{y} = -2n\dot{x}$$

$$\ddot{z} = -n^2z$$

The in-plane motion (x, y) remains coupled through Coriolis terms. The out-of-plane coordinate z decouples entirely into a simple harmonic oscillator at frequency n. This decoupling is exact at the linear level and remains a good approximation as long as the separation stays small. The linearization introduces a relative error on the order of $||r||/R_0$, which for the sub-kilometre scenarios examined here is below 10^{-5}.

4.2 State-Space Form

With state vector $X = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$ the CW equations take the first-order form $\dot{X} = AX$, where:

```
A = [ 0     0     0     1     0     0  ]
    [ 0     0     0     0     1     0  ]
    [ 0     0     0     0     0     1  ]
    [ 3n²   0     0     0    2n     0  ]
    [ 0     0     0   -2n     0     0  ]
    [ 0     0   -n²    0     0     0  ]
```

Fig. 2 — Linearized CW system matrix A. The 3n² entry captures the gravity-gradient effect; ±2n are Coriolis; -n² is the out-of-plane restoring term.

The top three rows couple position to velocities through the identity block. The gravity-gradient term $3n^2$ in the fourth row couples radial displacement back to radial acceleration, generating the characteristic elliptic in-plane modes. There is no restoring term in the y equation, which is the structural reason for along-track drift under generic initial conditions.

05 EIGENSTRUCTURE AND STABILITY ANALYSIS

5.1 Eigenvalue Computation

The eigenvalues of A are computed numerically by eig(A) and verified analytically by exploiting the block structure. Re-ordering states as $[x, \dot{x}, y, \dot{y} \mid z, \dot{z}]^T$ decouples the characteristic polynomial into an in-plane factor and an out-of-plane factor. The out-of-plane 2×2 block has characteristic equation $\lambda^2 + n^2 = 0$, giving a conjugate pair at $\pm jn$. The in-plane 4×4 block factors as $\lambda^2(\lambda^2 + n^2) = 0$, yielding a double zero and another conjugate pair at $\pm jn$.

Eigenvalue	Value	Type
$\lambda_{1,2}$	0 (double)	Double zero — Jordan bloc
$\lambda_{3,4}$	$\pm jn = \pm j\ 1.1067\times10^{-3}$	Purely imaginary pair
$\lambda_{5,6}$	$\pm jn = \pm j\ 1.1067\times10^{-3}$	Purely imaginary pair

Table 2 — Complete eigenvalue spectrum of the CW system matrix A.

All six eigenvalues lie on the imaginary axis, so the system is marginally stable: bounded but not asymptotically stable. The critical feature is the double zero eigenvalue. Since rank(A) = 4, the null space is two-dimensional and the Jordan form of A contains a 2×2 Jordan block for $\lambda = 0$. The state-transition matrix e^{At} therefore contains terms proportional to t, generating secular drift in the along-track direction for any initial condition that excites the null-space Jordan mode.

The physical consequence is immediate: without active control or careful initial-condition selection, the chaser will drift indefinitely along the -y axis. The MATLAB script verifies this by printing rank(A) and the rank defect to the console, and flags the presence of zero eigenvalues with a warning about secular drift.

5.2 Physical Interpretation of the Modal Structure

Three distinct dynamical modes govern the free response. The double zero mode contributes a constant offset and a term growing linearly with time. Projecting the initial state onto this mode, the coefficient of the secular term is $C = \dot{y}_0 + 2nx_0$. This quantity vanishes if and only if $\dot{y}_0 = -2nx_0$, the classical CW closed-orbit condition. When $C = 0$, the trajectory is permanently bounded; when $C \neq 0$, the chaser drifts backward along the y-axis at rate -3C km per orbit.

The $\pm jn$ in-plane mode drives the characteristic 2:1 CW ellipse: radial amplitude x_0 maps to along-track amplitude $2x_0$. The ±jn out-of-plane mode is a pure harmonic oscillator at frequency n with conserved amplitude $\sqrt{(z_0^2 + (\dot{z}_0/n)^2)}$, completely decoupled from the in-plane dynamics. This decoupling is an exact consequence of the linear structure and persists at the analytical solution level.

Scenario A is the canonical demonstration of the closed-orbit condition: constructed with $\dot{y}_0 = -2nx_0$, it traces a regular 2:1 ellipse with no secular growth. All other scenarios have $C \neq 0$ and drift, with Scenarios B and D exhibiting the pure drift case where the oscillatory in-plane terms cancel identically.

06 NUMERICAL COMPARISON: LINEAR VS NONLINEAR

6.1 Linear Free Response

The linearized CW equations are integrated under identical conditions to the nonlinear model: same ode45 tolerances, same five initial conditions, same time interval [0, 2T]. The only difference is the flag = 1 argument passed to odefun, which substitutes the exact r^{c-3} terms with their first-order Taylor expansion. Results are stored in t_lin{k} and res_lin{k} for subsequent comparison.

The function plot_trajectories_comparison overlays the nonlinear (solid) and linear (dashed) 3D trajectories per scenario. For Scenarios A, C, and E, which satisfy the closed-orbit condition, the linear trajectories closely match the nonlinear ones, tracing the same 2:1 ellipses and out-of-plane sinusoids. For Scenarios B and D, both models produce straight backward-drifting paths that are visually indistinguishable at the scenario scale, though the linearization error analysis reveals small but measurable differences.

Scenario E presents the most visible discrepancy: by the second orbit the two trajectories have dephased measurably, with the linear CW ellipse gradually diverging from the nonlinear path. This is the expected behavior at large separations, where the neglected second-order terms are no longer negligible.

6.2 Error Analysis and Linearization Validity

The position error is defined as $e(t) = r_{nl}(t) - r_{lin}(t)$. Since ode45 uses adaptive time-stepping, the nonlinear solution is interpolated onto the linear time grid via pchip interpolation before subtraction. The error norm $||e(t)||$ and its three components are computed and plotted for each scenario.

Scenario	max $||e||$ [km]	$||r_0||/R_0$	D
A	2.70×10^{-6}	1.45×10^{-5}	Oscillatory (
B	2.60×10^{-4}	1.45×10^{-5}	Along-track
C	4.11×10^{-5}	1.61×10^{-5}	Out-of-plane
D	2.59×10^{-4}	1.60×10^{-5}	Along-track
E	5.47×10^{-1}	2.19×10^{-3}	All compone

Table 3 — Maximum position error norm and relative initial displacement for each scenario. The scaling $||e||\_{max} \propto ||r_0||^2/R_0$ *is consistent throughout.*

For Scenarios A through D the initial displacements are 0.0003-0.3 m ($||r_0||/R_0 \approx 10^{-5}$), producing second-order remainders of order 10^{-6} km — entirely consistent with the observed errors. The CW linearization is therefore an accurate proxy for all four scenarios over the full two-orbit window. Scenario E is qualitatively different: its initial displacement of approximately 15.8 km drives the error to the 10^{-1} km level, three orders of magnitude above the small-offset cases. The two trajectories gradually dephase over successive orbits, and higher-order corrections (Yamanaka-Ankersen or Tschauner-Hempel equations) would be required for reliable maneuver planning at this separation scale.

07 ANALYTICAL SOLUTION AND VALIDATION

7.1 Closed-Form CW Solution

The CW system $\dot{X} = AX$ is solved analytically by exploiting the decoupling between the in-plane and out-of-plane subsystems. The out-of-plane equation $\ddot{z} = -n^2 z$ is a simple harmonic oscillator with general solution:

$$z(t) = z_0 \cos(nt) + (\dot{z}_0/n) \sin(nt)$$

The in-plane solution is obtained by integrating the conserved quantity from $\ddot{y} = -2n\dot{x}$ and substituting into the $\ddot{x}$ equation to produce a forced oscillator. The general in-plane solution, valid for arbitrary initial conditions including non-zero $\dot{x}_0$, is:

$$x(t) = (4x_0 + 2\dot{y}_0/n) + (-3x_0 - 2\dot{y}_0/n) \cos(nt) + (\dot{x}_0/n) \sin(nt)$$

$$y(t) = y_0 - (2\dot{x}_0/n) + C\cdot t + (6x_0 + 4\dot{y}_0/n) \sin(nt) + (2\dot{x}_0/n) \cos(nt)$$

$$\text{where } C = -(6nx_0 + 3\dot{y}_0) = -3(\dot{y}_0 + 2nx_0)$$

The secular coefficient C is the time-domain signature of the double zero eigenvalue. It vanishes if and only if $\dot{y}_0 = -2nx_0$, the CW closed-orbit condition. For Scenario D, $C = 0.5nx_0 \neq 0$, and substituting the specific initial conditions shows that the oscillatory amplitudes in x and y cancel identically, leaving $x(t) = x_0 = 0.1$ km, $y(t) = -1.5nx_0\, t$, $z(t) = z_0 \cos(nt)$. The total along-track displacement over two orbital periods is $\Delta y = -6\pi x_0 \approx -1.885$ km.

The analytical solution is validated against the ode45 integration of the linearized model for Scenario D. The position error norm $||e(t)||$ remains below 10^{-6} km throughout the two-orbit window, confirming that the analytical expression is the exact closed-form solution of the linearized system and that the numerical integrator tracks the secular trend without spurious accumulation.

7.2 Scenario Characterization

Applying the closed-form expressions to all five scenarios reveals their distinct dynamical characters. Scenario A traces a closed 2:1 ellipse with an independent out-of-plane sinusoid, entirely bounded. Scenario B produces a constant radial offset at x_0 with pure linear along-track drift; the oscillatory terms cancel because the initial velocity is insufficient to excite the ±jn mode. Scenario C isolates the out-of-plane mode: x

= 0, $y = y_0$ = constant, $z(t) = z_0 \cos(nt)$. Scenario D combines B’s drift with an independent out-of-plane oscillation. Scenario E is structurally identical to A but at 150 times the scale, producing a large-amplitude bounded orbit where the CW approximation begins to break down.

08 CLOSING MANEUVER DESIGN

8.1 STM-Based Two-Impulse Targeting

The closing maneuver transfers the chaser from the drifting Orbit D to the target parking Orbit A via two impulsive burns: an injection burn ΔV_1 applied after waiting time $t_a^{\text{aæt}}$ on Orbit D, and an arrival burn ΔV_2 applied after a free-flight arc of duration TOF. The closed-form CW state-transition matrix (STM) partitions as:

$$\Phi(t) = \begin{bmatrix} \Phi_{rr}(t) & \Phi_{rv}(t) \\ \Phi_{vr}(t) & \Phi_{vv}(t) \end{bmatrix}$$

where the four 3×3 blocks map initial positions to final positions, initial velocities to final positions, and so on. The analytical expressions for each block are given by the Laplace inverse of $(sI - A)^{-1}$, evaluated once in get_STM_blocks(t, n) and reused throughout. The targeting procedure from Fehse [1] proceeds as follows: the target’s arrival state on Orbit A is computed by propagating the initial conditions forward by τ_A; the chaser’s departure state is computed by propagating Orbit D forward by $t_a^{\text{aæt}}$; the required post-burn velocity $v_r^{e\psi}$ is obtained by inverting $\Phi_{rv}(\text{TOF})$; and the two burns follow directly.

A third optimization variable τ_A, the phase of Orbit A at the rendezvous point, decouples the spatial target from the total mission time. This additional degree of freedom allows the optimizer to find geometrically favorable near-tangent arcs, analogous to Hohmann transfers, that are inaccessible in a fixed-time two-variable formulation.

8.2 Optimization Setup and Solver Comparison

The optimization vector is X_opt = [$t_a^{\text{aæt}}$, TOF, τ_A] with constraints $t_a^{\text{aæt}} \geq T/3$, TOF ≤ 0.99T, τ_A ∈ [0, T], and per-component burn magnitude ≤ 10 m/s. The cost function $||e|| = ||\Delta V_1|| + ||\Delta V_2||$ is scaled by 10^6 to prevent gradient-based solvers from terminating on an apparently flat objective landscape. Orbital singularities where $\Phi_{rv}(\text{TOF})$ becomes nearly singular are detected via rcond and penalized.

Four solvers were compared on the same cost function and constraints to identify the global minimum:

Solver	t_wait [s]	TOF [s]	τ_A [s]
fmincon (local)	1892	3505	2808
GlobalSearch	4094	5504	5451

ga (genetic)	4027	4673	4576	0.1731	2.25
particleswarm	4094	5505	5452	0.1478	0.27

Table 4 — Solver comparison for the three-parameter fuel-optimal transfer. GlobalSearch and particleswarm reach the global minimum; fmincon alone is trapped at a local one.

The local solver fmincon converges rapidly to the nearest feasible point, which happens to be a suboptimal geometry with $\Delta V = 0.2243$ m/s. GlobalSearch combines systematic scatter-search exploration with an SQP final polish, finding the global minimum at $\Delta V = 0.1478$ m/s — a 34 percent reduction. The genetic algorithm avoids the local trap but lacks the precision needed for accurate constraint enforcement. Particleswarm matches GlobalSearch in cost but does not natively support strict nonlinear constraints. GlobalSearch is therefore adopted as the baseline solver.

The key physical insight from the global solution is that the τ_A freedom allows the chaser to wait until the orbital geometry produces a near-tangent intercept: the chaser drifts on Orbit D for 0.721T, executes a long near-tangential transfer arc of duration 0.970T, and arrives at the rendezvous point after a total mission time of 1.691T. Nonlinear verification of the GlobalSearch burns on the full nonlinear dynamics shows a residual position error of 2.12 cm and residual velocity error of 0.0000 m/s, confirming that the CW transfer design is fully consistent with the exact dynamics at this separation scale.

8.3 Time-Optimal Transfer Variants

Two time-optimal variants were explored by replacing the fuel objective with $J = (t_{a}^{a\text{æ}t} + TOF)/T$, with the per-component burn limit serving as the active constraint. With a high-thrust limit of 10 m/s per component, the optimizer finds a nearly direct arc: TOF = 12.56 s (0.002T), total mission time 1904.89 s (0.336T), fuel cost 31.38 m/s. With a low-thrust limit of 0.5 m/s per component, the geometric constraint forces a longer arc: TOF = 282 s (0.050T), total time 2174.40 s (0.383T), fuel cost 1.50 m/s. These two cases illustrate the fundamental time-fuel trade-off in impulsive orbital mechanics: reducing the transfer time requires saturating the thrust limit, at a fuel cost three to twenty times higher than the fuel-optimal baseline.

09 LQR CONTROL LAW FOR FINAL APPROACH

9.1 System Controllability and LQR Design

The final approach phase is governed by the augmented CW system $\dot{X} = AX + BU$, where $B = [0_{3x}{}^{3}; I_3]$ maps control accelerations $U = [a\_x, a\_y, a\_z]^T$ directly into the velocity states. Controllability is verified by evaluating the controllability matrix $C = [B, AB, A^2B, A^3B, A^4B, A^5B]$ via ctrb(A,B); rank(C) = 6 confirms that the system is fully controllable.

The adopted control law is full-state linear feedback $U = -KX$, giving closed-loop dynamics $\dot{X} = (A - BK)X$. The gain matrix K is computed by the Linear Quadratic Regulator (LQR), which minimizes the infinite-horizon quadratic cost:

$$J = \int_0^\infty (X^T Q X + \rho U^T R\_b U)\, dt$$

The weighting matrices are constructed by Bryson’s rule [4]: $Q = \mathrm{diag}(1/r^{m}{}_{a}$₥$^2 I_3, 1/v^{m}{}_{a}$₥$^2 I_3)$ and $R\_b = (1/a^{m}{}_{a}$₥$^2) I_3$, with $r^{m}{}_{a}$₥ = $||r_0||$, $v^{m}{}_{a}$₥ = $n||r_0||$, and $a^{m}{}_{a}$₥ = 5×10^{-6} km/s². This normalization ensures that each integrand term is O(1) when the corresponding state is at its tolerance scale. The scalar weight ρ is the single remaining design degree of freedom: large ρ penalizes control heavily, producing a gentle fuel-saving approach; small ρ rewards fast error nulling at the cost of higher accelerations.

9.2 Weight Optimization and Nonlinear Validation

Rather than scanning a grid of ρ values, the fuel-optimal weight is found directly via scalar constrained minimization:

```
ρ_opt = fmincon(@LQR_cost_fun, ρ₀, ...,
@LQR_nonlcon, ...)
```

Fig. 3 — Scalar rho optimization. LQR_cost_fun evaluates the closed-loop delta-V via full nonlinear ode45 simulation; LQR_nonlcon enforces docking within 1.1T.

The cost function LQR_cost_fun evaluates $\Delta V = \int||U(t)||\_2\, dt$ by running a full nonlinear ode45 simulation with the docking event function. The constraint function LQR_nonlcon checks whether the 1-metre docking sphere is reached within 1.1T; if not, a penalty proportional to the final miss distance is substituted. To avoid redundant integrations when fmincon evaluates cost and constraint at the same ρ, both functions route through the cached_ode helper, which stores the last ode45 result and returns it directly on repeated calls.

The optimal weight ρ_opt = 650.37 produces a docking maneuver in which position and velocity errors converge monotonically to zero within one orbital period. The gain K computed from lqr(A, B, Q, ρ_opt R_b) is synthesized on the linearized plant but applied without modification to the full nonlinear dynamics, exploiting the fact that at sub-kilometre separations the neglected second-order terms are negligible relative to the dominant accelerations — consistent with the linearization error analysis of Section 06.

10 FULL MISSION SIMULATION

10.1 Four-Phase Simulink Integration

The complete mission is simulated in Simulink over a total duration of 5T, with the four mission phases combined into a single continuous simulation run. The Simulink model integrates the nonlinear relative dynamics using standard mathematical blocks feeding into Integrator blocks for positions and velocities.

Phase A (Homing) begins at t = 0 with the chaser on Orbit D. No control is applied; the chaser drifts under natural orbital mechanics until $t = t_a^{\text{aæt}}$. Phase B (Closing) implements the two-impulse transfer by triggering instantaneous velocity jumps via External Reset ports and Initial Condition blocks on the velocity integrators. State Ports read the pre-burn velocity to compute the burns without introducing algebraic loops. Phase C (Station Keeping) covers the interval $[t_a^{\text{aæt}} + \text{TOF}, t_a^{\text{aæt}} + \text{TOF} + 0.5T]$: no control is applied and the chaser traces the natural closed relative orbit of Scenario A around the target. Phase D (Final Approach) activates the LQR feedback law U = -KX at $t\_dock = t_a^{\text{aæt}} + \text{TOF} + 0.5T$ via a timing switch; the continuous acceleration drives the relative state to zero.

10.2 Delta-V Budget

The total delta-V budget separates the impulsive component from the two-impulse transfer and the continuous component from the LQR final approach.

Mission Phase	Maneuver Type	
Phase B: Injection burn	Impulsive (ΔV_1)	0.
Phase B: Arrival burn	Impulsive (ΔV_2)	0.
Phase D: Final approach	Continuous ($\int \|\|U\|\| dt$)	0.
Total mission	Combined budget	0.

Table 5 — Total delta-V budget by mission phase. Impulsive costs from the GlobalSearch transfer; continuous cost integrated from the Simulink output.

The continuous LQR cost dominates the total budget at 66 percent of the total. The impulsive injection burn represents the largest single impulse, consistent with the near-tangential geometry of the GlobalSearch solution. The cumulative delta-V profile is computed in the MATLAB post-processing step by reconstructing U(t) = -KX(t) sample-by-sample from the Simulink state output and integrating with cumtrapz.

11 GLOBAL MISSION CO-OPTIMIZATION

The sequential design approach — optimizing the transfer geometry in Section 08 independently of the controller design in Section 09 — produces a locally consistent but globally suboptimal solution. A transfer geometry that minimizes impulsive fuel may place the chaser at a parking-orbit phase that requires an aggressive, fuel-heavy LQR correction, and vice versa.

A global co-optimization was therefore carried out that jointly selects the transfer geometry and controller tuning to minimize the total mission delta-V. The optimization vector is expanded to four dimensions X_opt = [$t_{a}^{\text{aæt}}$, TOF, τ_A, ρ], and the objective combines impulsive and continuous costs:

```
J = ||ΔV₁|| + ||ΔV₂|| +
∫_{t_dock}^{t_final} ||U_{cl}(t)||_2 dt
```

A penalty term J + ||r_final|| × 10^{12} is added when the docking event is not triggered within the time budget. Particle swarm optimization (particleswarm) was selected as the solver because it is a derivative-free global heuristic that evaluates the cost directly without computing gradients, and naturally handles the non-smooth landscape introduced by the event logic and penalty terms. The search domain is $t_{a}^{\text{aæt}}$ ∈ [T/3, 3T], TOF ∈ [0.001T, 0.99T], τ_A ∈ [0.001T, 0.99T], ρ ∈ [0.1, 10000].

Parameter	Sequential Design	Co-Optimized	Savings
t_wait	4093.66 s (0.721T)	3558.31 s (0.627T)	—
TOF	5504.24 s (0.970T)	4842.93 s (0.853T)	—
τ_A	5451.05 s (0.960T)	4623.13 s (0.815T)	—
LQR weight ρ	650.37	6392.04	—
Injection ΔV	0.1028 m/s	0.1080 m/s	+5%
Arrival ΔV	0.0450 m/s	0.0546 m/s	+21%
Final approach ΔV	0.2813 m/s	0.1648 m/s	-41%
Total mission ΔV	0.4292 m/s	0.3274 m/s	-24%

Table 6 — Comparison of sequential and co-optimized mission designs. The global optimizer jointly tunes transfer geometry and LQR weight, reducing total ΔV by 24 percent.

The co-optimization reduces total mission delta-V by approximately 24 percent relative to the sequentially designed baseline, from 0.4292 to 0.3274 m/s. The largest saving comes from the Phase D continuous cost, which falls by 41 percent. The co-optimizer achieves this by selecting a higher LQR weight ρ = 6392 (compared to 650 from the sequential design), making the controller gentler and more fuel-efficient at the cost of a longer approach time. This is possible precisely because the co-optimizer simultaneously selects a transfer geometry that places the chaser at a more favorable parking-orbit phase, allowing the gentler controller to still satisfy the docking-time constraint. The impulsive costs increase slightly because the co-optimizer accepts a slightly less fuel-efficient transfer arc in exchange for the large saving in the continuous phase.

12 CONCLUSIONS

This paper has presented a complete end-to-end engineering treatment of the spacecraft proximity operations problem, from orbital mechanics fundamentals through globally optimal control design. The principal technical contributions and findings are summarized as follows.

The Clohessy-Wiltshire linearization was derived from first principles and shown to produce position errors below 10^{-4} km over two orbital periods for separation scales up to 300 m, scaling consistently with the second-order remainder $||r||/R_0$. At the 15.8 km separation of Scenario E, the linearization error reaches 10^{-1} km, identifying the practical boundary of CW validity.

The double zero eigenvalue of the system matrix A was identified as the structural cause of secular along-track drift, manifesting as a size-2 Jordan block in the state-transition matrix. The closed-orbit condition $\dot{y}_0 = -2nx_0$ was derived analytically as the necessary and sufficient condition for the initial state to project entirely onto the oscillatory modes, eliminating the secular term.

The globally optimal two-impulse transfer was designed using the CW STM and a three-parameter optimization. Comparing four solvers established that local gradient-based methods are unreliable on this non-convex problem: fmincon was trapped at a local minimum costing 0.2243 m/s, while GlobalSearch and particleswarm found the global minimum at 0.1478 m/s, a 34 percent reduction achieved by exploiting the τ_A degree of freedom to find a near-tangent transfer geometry.

The LQR final approach was designed with Bryson-rule weighting and a scalar rho optimization that finds the fuel-optimal controller weight without scanning a parameter grid. The controller was validated on the full nonlinear plant, confirming that the linearization-based design is fully consistent with the exact dynamics at sub-kilometre separation.

The global co-optimization, which jointly optimizes the transfer geometry and the LQR weight via particle swarm, reduced total mission delta-V by 24 percent relative to the sequentially designed baseline, from 0.4292 to 0.3274 m/s. The key mechanism is the simultaneous selection of a gentler controller (higher ρ) and a transfer geometry that places the chaser at a phase that allows the gentler approach to still satisfy the time constraint.

Future work could extend the analysis to non-circular target orbits using the Tschauner-Hempel equations, incorporate J2 and atmospheric drag perturbations via a mean-motion correction, and replace the impulsive transfer model with a continuous low-thrust arc using Pontryagin’s minimum principle.

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The following references were consulted in the preparation of this work. Items [1] and [2] form the primary theoretical basis; items [3] through [6] provide the underlying orbital mechanics and control theory; items [7] through [10] cover the numerical methods and software tools used in the implementation.

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